MATH 320 NOTES, WEEK 9

2.3 Composition of linear transformations

Recall that:

Lemma 1. *Suppose that V, W and Z are vector spaces over F and that $T : V \to W$ and $U : W \to Z$ are linear transformations. Then the composition $UT : V \to Z$ defined by*

$$UT(x) = U(T(x))$$

is also a linear transformation.

Also, recall that last week we showed

$$(\dagger) : [T]^\beta_\alpha [x]_\alpha = [T(x)]_\beta,$$

whenever $T : V \to W$ is a linear transformation and α, β are finite bases for V and W, respectively.

Example: Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ and $D : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be given by:

- $T(p) = \int_0^x p(t)dt$, and
- $D(p) = p'$.

Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3\}$. Then

$$[D]^\beta_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \text{ and } [T]^\alpha_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Note that $[D]^\beta_\alpha [T]^\alpha_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 = [I]_\alpha$, where $I(p) = p$ is the identity linear transformation.

Next we talk about matrix representation of compositions of linear transformations.

Theorem 2. *Suppose that V, W and Z are finite dimensional vector spaces over F and that $T : V \to W$ and $U : W \to Z$ are linear transformations. Suppose α, β, γ are bases for V, W, and Z, respectively. Then*

$$[UT]^\gamma_\alpha = [U]^\gamma_\beta [T]^\beta_\alpha.$$

Proof. Let $\dim(V) = n, \dim(W) = k, \dim(Z) = m$, and $\alpha = \{x_1, ..., x_n\}$.

Let $A = [UT]^\gamma_\alpha$, $B = [U]^\gamma_\beta$ and $C = [T]^\beta_\alpha$. Note that A is $m \times n$, B is $m \times k$, and C is $k \times n$.

First we show that for each $b \in F^n$, $Ab = BCb$.

Let $b = \langle b_1, ..., b_n \rangle$ and $x = b_1x_1 + ... + b_nx_n$. Then $[x]_\alpha = \vec{b}$, and by $(\dagger)$, we have:

- $A\vec{b} = [UT]^\gamma_\alpha [x]_\alpha = [UT(x)]_\gamma$,
- $BC\vec{b} = [U]^\gamma_\beta [T]^\beta_\alpha [x]_\alpha = [U]^\gamma_\beta [T(x)]_\beta = [U(T(x))]_\gamma = [UT(x)]_\gamma$.

It follows that for every $b \in F^n$, $Ab = BCb$. In particular for every $1 \leq i \leq n$,

$$\text{the i-th column of } A = Ae_i = (BC)e_i = \text{the i-th column of } BC.$$

So, $A = BC$. □

Next we combine that addition of linear transformation and composition:

Some facts:

- $U(T_1 + T_2) = UT_1 + UT_2$,
- $(L_1 + L_2)U = L_1U + L_2U$,
- $IU = U = UI$, where I is the identity linear transformation i.e. $I(x) = x$ for all x.

Recall also, that if $T, L : V \to W$ are linear transformations and α, β and finite bases for V and W, respectively, then $[T + U]^\beta_\alpha = [T]^\beta_\alpha + [U]^\beta_\alpha$. In the same spirit as above, here is what we have:

Some facts about matrices:

- $A(B_1 + B_2) = AB_1 + AB_2$,
- $(C_1 + C_2)A = C_1A + C_2A$,
- $I_nA = A = AI_k$, where I_n is the identity $n \times n$ matrix, and A is $n \times k$.

2.4 Invertibility and isomorphism

In this section we will define when two vector spaces are isomorphic.

Definition 3. *Suppose V and W are vector spaces over a field F and $T : V \to W$ be a linear transformation. T is an* **isomorphism** *between V and W, if T is one to one and onto. When such a T exists, we say that V and W are isomorphic and write $V \cong W$.*

As our first key example, we have the following lemma:

Lemma 4. *Let V be a finite dimensional vector space over F with* $\dim(V) = n$. *Then*

$$V \cong F^n.$$

Proof. Let β be any basis for V. Then, as we have already seen, $\phi_\beta : V \to F^n$ given by

$$\phi_\beta(x) = [x]_\beta$$

is a one-to-one, onto linear transformation. I.e. ϕ_β is an isomorphism. □

Examples:

(1) $P_2(\mathbb{R}) \cong \mathbb{R}^3$,
(2) $P_n(F) \cong F^{n+1}$,
(3) $M_{k,n}(F) \cong F^{nk}$,
(4) If V and W are vector space over F, such that $\dim(V) = n, \dim(W) = k$, then $\mathcal{L}(V, W) \cong M_{k,n}(F) \cong F^{nk}$.

Note that if $V \cong W$ and $W \cong Z$, then $V \cong Z$. This is used in the last example above.

We also give one example with infinite dimension:

Example: Let $W = \{p \in P(F) \mid p(x) = a_1x+a_2x^2+...a_nx^n$ for some $n \geq 1\} = Span(\{x, x^2, x^3, ..., x^n, ...\})$. Then $P(F) \cong W$.

To show this, let $T : P(F) \to W$ be defined as follows. For a polynomial $p(x) = a_0 + a_1x + a_2x^2 + ...a_nx^n \in P(F)$, set

$$T(p) = a_1x + a_2x^2 + ... + a_nx^n.$$

Then T is an isomorphism.

Next we show that any isomorphism $T : V \to W$ has an **inverse** i.e. a linear transformation $T^{-1} : W \to V$, such that for all $x \in V$ and $y \in W$,

$$T^{-1}T(x) = x, \text{ and } TT^{-1}(y) = y$$

i.e. $T^{-1}T = I_V$ and $TT^{-1} = I_W$, the identity maps on V and W, respectively.

Lemma 5. *Suppose that $T : V \to W$ is a linear transformation. Then T is one-to-one and onto iff T has an inverse.*

Proof. For the first direction, suppose that T is one-to-one and onto. Then for every $y \in W$, since T is onto, there exists some $x \in V$ such that $T(x) = y$. Moreover, since T is one-to-one, this x is unique. Set $T^{-1}(y) = x$. Then by definition, for every $x \in V$, $T^{-1}T(x) = x$.

For the other direction, suppose that $T^{-1} : W \to V$ exists. First we show that T is one-to-one: let $x \in V$ be such that $T(x) = \vec{0}$. Then by definition of the inverse, $T^{-1}(\vec{0}) = x$. But also, since T^{-1} is linear, we have that $T^{-1}(\vec{0}) = \vec{0}$. Since T^{-1} is a function, we have that $x = \vec{0}$.

Next, to show that T is onto, take any arbitrary $y \in W$. Let $x = T^{-1}(y)$. Then by definition of the inverse, $T(x) = y$, and so $y \in \text{ran}(T)$.

□

Definition 6. *An $n \times n$ matrix A is invertible iff there is an $n \times n$ matrix B, such that $AB = BA = I_n$. The matrix B is called* **the inverse** *of A and is usually denoted by A^{-1}.*

Lemma 7. *An n by n matrix A is invertible iff L_A is invertible.*

Proof. If A is invertible, let B be its inverse. Then, if $Ax = \vec{0}$, since $BA = I_n$, we have that

$$x = I_n x = BAx = B\vec{0} = \vec{0}.$$

So $\ker(A) = \{\vec{0}\}$, and L_A is one-to-one. Also, for any $\vec{y} \in F^n$, let $\vec{x} = B\vec{y}$. Then

$$A\vec{x} = AB\vec{y} = I_n\vec{y} = \vec{y}.$$

So L_A is onto. It follows that L_A is invertible.

On the other hand, if L_A is invertible, let $T : F^n \to F^n$ be its inverse i.e. for all $x \in F^n$, $L_A T(x) = TL_A(x) = x$. Let e be the standard basis for F^n, and let $B = [T]_\alpha$. Then for any $x \in F^n$,

$$ABx = [L_A]_\alpha [T]_\alpha x = [L_A T]_\alpha [x]_\alpha = [L_A T(x)]_\alpha = [x]_\alpha = x = I_n x.$$

So, $AB = I_n$. (Similarly, $BA = I_n$.) Then A is invertible and B is its inverse.

□

With a similar proof, we get that:

Lemma 8. *Let* $\dim(V) = \dim(W) = n$*, and* $T : V \to W$ *is a linear transformation,* β *any basis for* V*,* γ *any basis for* W*. Then* T *is invertible iff* $[T]_\beta^\gamma$ *is invertible.*

Moreover, in the case where T *is invertible, setting* $A = [T]_\beta^\gamma$*, and* $B = [T^{-1}]_\gamma^\beta$*, we have that* $B = A^{-1}$*.*